CHAPTER 16

# A neo-Kantian philosophy of mathematics

## 1 Historical outline

The nineteenth century saw the ascendancy of Kantian philosophy, which was still in a dominant position when the debate of the 1890s over the adoption of the Axiom of Completeness reached its climax, and set theory was born. As I shall show, Cantor, Frege and Hilbert were Kantians even though the general direction of their work did more than a little to undermine Kantian philosophy. By 1900 the tide of positivism was overwhelming, and a popular desire to overturn Kantianism, and its late nineteenth century scions of idealism, began to affect philosophers and mathematicians alike. A contemporary student now starting in the philosophy of mathematics soon discovers that the main schools are: formalism, intuitionism and logicism, but even though the founders of each of these movements were all Kantians, the Kantian philosophy of mathematics has ceased to appear as such. As the twentieth century progressed, each of these philosophies evolved into new forms that bore less and less resemblance to their Kantian origins. Logicism floundered on technical difficulties, formalism evolved in the philosophy of formal systems, and intuitionism evolved into constructivism and even into strict finitism. As constructivism and finitism are consistent with modern formalism, it may be said that at the present time formalism is the only philosophy of mathematics left standing, and that it dominates contemporary debate. What is missing from the dialectic is a developed neo-Kantian philosophy.

## 2 The Kantian philosophy

Kant argues that as human beings we all possess an unconscious mental faculty that he calls imagination. The function of imagination is to render experience intelligible to us:

> ... the order and regularity in the appearances, which we entitle *nature*, we ourselves introduce. We could never find them in appearances, had not we ourselves, or the nature of our mind, originally set them there. (Kant [1781] Critique of Pure Reason, A. 123.p.146 Transcendental Deudction – the Relation of the Understanding in General, and the Possibility of Knowing them A Priori.]

Kant seeks to explain the origin of the necessity and certainty that we encounter with respect to logic and mathematics, which he regards as truths *synthetic a priori*. Such truths embody substantive knowledge concerning reality – principles that constrain real objects to obey them, and yet the knowledge of these truths cannot be derived from particular experience.

> A pure imagination[1], which conditions all *a priori* knowledge, is thus one of the fundamental faculties of the human soul. By its means we bring the manifold of intuition on the one side, into connection with the condition of the necessary unity of pure apperception on the other. The two extremes, namely sensibility and understanding, must stand in necessary connection with each other through the mediation of this transcendental function of imagination, because otherwise the former though indeed yielding appearances, would supply no objects of empirical knowledge, and consequently no experience. Actual experience, which is constituted by apprehension, association (reproduction), and finally recognition of appearances, contains in recognition, the last and highest of these merely empirical elements of experience, certain concepts which render possible the formal unity of experience, and therewith all objective validity (truth) of empirical knowledge. These grounds of the recognition of the manifold, so far as they concern *solely the form of an experience in general*, are the *categories*. ... Only by means of these fundamental concepts [categories] can appearances belong to knowledge or even to our consciousness, and so to ourselves. (Kant [1781], Critique of Pure Reason, A. 123.p.146 Transcendental Deduction – the Relation of the Understanding in General, and the Possibility of Knowing them A Priori. Italics are due to Kant.)

The imagination unconsciously structures experience for us by means of "fundamental concepts" that Kant calls *categories*. Each category corresponds to some power of the unconscious mind coordinated by imagination to render experience intelligible. The categories originate ultimately as *powers of the unconscious mind to sort, order, interpret, and make sense of our experience*. Categories manifest themselves to us in consciousness, *after the fact*, as *concepts*. We perceive these categories as immutable, permanent principles of the organisation of our experience, and regard them as *fundamental concepts*, and in epistemology as *primitive propositions*. We *feel* that we cannot question their certainty.

[1] The use of *imagination* here is specific to Kant. It does not mean what Hume calls *fancy*, which is the creative ability of the mind to make new combinations out of old. To focus on the definition that Kant offers in this passage, imagination is the power to "bring the manifold of intuition on the one side, into connection with the condition of the necessity unity of pure apperception on the other." It welds all experience into one coherent unity. We might say that its fundamental and abiding message to the self, whatever its relative state of consciousness, is "experience makes sense".

# 3 Logicism

In the *Foundations of Arithmetic* (Frege [1980], p. 101 et seq.) Frege writes: -

> I have no wish to incur the reproach of picking petty quarrels with a genius [Kant] to whom we must all look up with grateful awe; I feel bound, therefore, to call attention also to the extent of my agreement with him, which far exceeds any disagreement. To touch upon what is immediately relevant, I consider KANT did great service in drawing the distinction between synthetic and analytic judgements. In calling the truths of geometry synthetic and a priori, he revealed their true nature. And this is still worth repeating, since even to-day it is often not recognized. If KANT was wrong about arithmetic, that does not seriously detract, in my opinion, from the value of his work. His point was, that there are such things as synthetic judgements a priori; whether they are to be found in geometry only, or in arithmetic as well, is of less importance.

This quotation confirms that Frege adopted the Kantian position with regard to geometry, which he accepts as *synthetic a priori.* However, he reclassifies arithmetic, *using the Kantian scheme*, as *analytic a priori*, being derived from the law of non-contradiction within logic; he does not appear to discuss whether the law of contradiction does itself require a synthetic a priori basis. Frege is at pains to deny psychologism as a foundation of mathematical truth, and the manner in which he does so runs contrary to the *spirit of Kantian philosophy.* For Kant all knowledge whatsoever is concerned with our psychology, though not as individuals per se but as rational, conscious beings who are members of a certain species. No-one who embraces the spirit of Kantian philosophy could write the following two sentences: "Mathematics is not concerned with the nature of our mind, and the answer to any question whatsoever in psychology must be for mathematics a matter of complete indifference." [*Ibid.* p. 105] "Time is only a psychological necessity for numbering, it has nothing to do with the concept of number." [*Ibid.* p. 53.] While it appears that Kantian philosophy formed the framework of Frege's early philosophical training, he was by instinct drawn in a different direction, and *never fully comprehended Kant.*

Frege conceived his programme in Kantian terms: to reclassify arithmetic from synthetic a priori to analytic a priori; specifically, the reclassification of number as a concept derived from the logic of Aristotle. Aristotle's logic is based on the intensional distinction of subject and predicate, and although Frege needed to rewrite and expand Aristotean logic, he retained intensions: -

1. His logic is called *concept writing.*
2. He replaces the subject-predicate distinction by the distinction between propositional function and argument.

3. He adopts the “no-class” theory. Classes[2] are defined by class abstraction (comprehension) as the ranges of values (extensions) of concepts (that is, functions). Therefore, classes are an abbreviation or façon de parler for functions. All results about classes could be translated into results about functions.
4. (Cardinal) numbers are then defined as equivalence classes of classes under the function of equinumerosity.

So this remains a logic in which intensions are primitive, and even if the Fregean programme succeeds it merely transfers the weight of justification of arithmetic from the primitive concept of number to the primitive concept of intension. Overall, arithmetic remains synthetic a priori, but technically, and in relation to logic, it becomes an analytic extension of it. We have seen above that Frege made no attempt to reclassify geometry in this way, which he accepted was to remain synthetic a priori in its own right.

With the benefit of hindsight we see that this attempted reclassification was misguided, for number is simply too fundamental a category to be so analysed. Nonetheless, the Fregean programme was a worthy and in many respects fruitful attempt. Naive Platonism postulates a supra-sensible faculty of the mind to “perceive” or “conceive” of abstract objects. The Platonic object is problematic. Firstly, by conceiving the relation of mind to number as too closely analogous to the relation of perception to perceptual object, it makes number as form “mysterious”. Secondly, it fails to account for the applicability of mathematics. On this latter point Frege’s theory is particularly strong. For Fregean classes are built on the basis of individuals and therefore Fregean cardinal numbers embrace individuals. The theory embraces applications because the equivalence classes subsume individuals. Mathematics and science are exhibited in very close connection.

It is often said that it was Gödel’s incompleteness theorem that put paid to logicism, but logicism, as a philosophy of number, had failed long before that. Specifically, it failed the day (in 1902)[3] that Frege received Russell’s letter containing his famous paradox. [See 4.6 below] So what did Frege see that day that Russell laboured not to? What prevented him from ever writing again?

Frege’s theory claims that the relation of numerical equivalence *partitions* sets into subsets of numerically equivalent sets. This claim is contentious. Partitions may be defined on a set. For example, I can partition a finite set of natural numbers into sets of odd and even numbers. I start with a set, and then partition it by means of an equivalence relation defined on that set. So what is the “universal set” that I start with when I partition sets into subsets of numerically equivalent sets? The answer one would like to give is *the set of all sets*. But it follows from Russell’s paradox [4.6] that the concept of *the set of all sets* is self-contradictory and does not define a set. Benacerraf succinctly expresses this problem in the following

[2] In this context, *class* is a synonym of *set*, and is, like it, a *definite, extensional multiplicity*.

[3] In van Heijenoort [1967]

quotation in which he uses "class" as a synonym of "set": "In no consistent theory is there a class of all classes with seventeen members, at least not alongside the other standard set-theoretical apparatus."[4]

Russell attempted to solve the problem with a theory of types; but the problem is compounded in a set theory with types. Rather than excurse on this, let us conclude with the definitive statement of this problem from Paul Benacerraf.

> If numbers are sets, then they must be *particular sets*, for each set is some particular set. But if the number 3 is really one set rather than another, it must be possible to give some cogent reason for thinking so; for the position that this is an unknowable truth is hardly tenable. But there seems to be little to choose among the accounts. Relative to our purposes in giving an account of these matters, one will do as well as another, stylistic preferences aside. There is no way connected with the reference of number words that will allow us to choose among them, *for the accounts differ at places where there is no connection whatsoever between features of the accounts and our uses of the words in question.* If all the above is cogent, then there is little to conclude except that any feature of an account that identifies 3 with a set is a superfluous one – and that therefore 3, and its fellow numbers, could not be sets at all. (In Benacerraf and Putnam [1987] p. 285)

This is known in the literature as *Benacerraf's problem*. Let us look at Benacerraf's solution, which he calls his "way out".

1. "Numbers are not objects at all." (p.291)
2. "To *be* the number 3 is no more and no less than to be preceded by 2, 1, and possibly 0, and to be followed by 4, 5, and so forth." (p.291)
3. "Arithmetic is therefore the science that elaborates the abstract structure that all progressions have in common merely in virtue of being progressions." (p.291)
4. "… every analysis of number ever presented has had a recursive "less than" relation. If what we are generating is notation, the most natural way for generating it is by giving recursive rules for getting the next element from any element you may have." (p.293)
5. He describes himself as a "sort of formalist" (p.293).

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[4] . (In Benacerraf and Putnam [1987] p. 284) The term class is used in relation to set theory to allow for the possibility that there are collections that are not sets – a *proper class* is a collection that is not a set. [See Chap. 2 / 1.3.5] This is a *façon de parler* that allows us to refer to properties that are meaningful but self-contradictory if regarded as actually collecting objects together. However, class and set are also loosely used as synonyms, and it is in that sense that Benacerraf uses "class" in this quotation.

The term "abstract structure", which is primitive here, could be interpreted as an intension. However, Benacerraf's "way out" is a species of formalism in which number as object (and/or concept) is replaced by mechanical processes (that is, effective or recursive progressions) that could be implemented on a computer. Once again we see that a partially conceived Kantian solution to the problem of number, in the form of logicism, leads by a process of cultural-logical inevitability to its nemesis in formalism. In formalism the question of how number is related at all to concepts of the understanding (categories) is abandoned; if we have any concept at all our grasp of it is treated as an epiphenomenon. Only a fully-developed neo-Kantian solution to the problem of the foundation of mathematics will be immune from such a collapse into formalism.

## 4 Cantor, the Axiom of Completeness and Hilbertism

Cantor's expresses his preference for Kant in the following quotation from the *Grundlagen*, Section 5: -

> For in addition to or in place of the mechanical explanation of nature, (which has all the aids and advantages of mathematical analysis at its disposal, and yet the one-sidedness and inadequacy of which has been exposed so well by Kant), previously there has not been even one attempt to pursue this beginning [i.e. Kant's] armed with the same mathematical rigor for the purpose of reaching beyond that far-reaching *organic* explanation of nature." [Quoted in Dauben [1979] p.293].

In contrast to Frege and Hilbert, Cantor was an out-and-out opponent of positivism, and he sometimes included Kant as a target of his attack; for example in a letter to Valson, January 31, 1886: "... that the greatest achievement of a genius (like Newton) despite the subjective religiosity of its author, when it is now united with a true philosophical and historical spirit, leads to effects (and, I assert, necessarily does so) by which it seems highly questionable whether the good in them simultaneously conveyed to mankind won't be significantly surpassed by the bad. And the most detrimental (of these effects) it seems to me are the errors of the "positivism" of Newton, Kant, Comte and others upon which modern scepticism depends." [Quoted in Dauben [1979] p.295]. Cantor was a defender of the actual infinite, of strong Christian persuasion, and overall his philosophy of mathematics appears closer to classical Platonism than to the position of Kant, who represents a compromise between Rationalism (Platonism) and Empiricism.

Cantor's set theory employs the notion of the actual infinite; his concept of an ordinal involves *taking the limit of all the natural numbers*, representing these as a totality, the *first* infinite ordinal, $\omega$, and thereafter, *carrying on counting upwards*, $\omega, \omega+1, \omega+2, \ldots$ $\ldots \omega 2, \omega 2+1, \ldots$. The Greeks concluded that the concept of the actual infinite was a source of contradiction, whose difficulties were encapsulated in the paradoxes of Zeno. Aristotle legislated against the use of the actual infinite in all mathematical discourse; the potential infinite was justified, the actual not. The Greeks adopted Aristotle's solution and the rule against the actual infinite was adopted in principle. The genesis of the calculus flouted this rule. The actual infinite is implicit in our conception of the number line as *composed of points*.[5] A point is an idealised element of a line that has no magnitude. Therefore, *no finite number of points will compose a line*; the line is thereby conceptualised as a continuum (continuous ideal magnitude) composed of an infinite number of points. Thus the actual infinite is *reborn*. An infinitesimal is a minute quantity of non-zero magnitude and yet incommensurable with a zero.[6] Infinitesimals were introduced into the foundation of calculus by Leibniz and implicitly by Newton in the form of *fluxions*; he wanted to eliminate them in favour of ratios, but did not succeed. Despite Berkeley's scathing attack in *The Analyst* (Berkeley, [1734]) on Newton's "evanescent quantities" most practising mathematicians of the day simply ignored the problem – they did not want to be forced out of the *paradise* Newton had created for them. Gauss famously endorsed the Aristotelan-Kantian rule against the actual infinite.

> But concerning your poof, I protest above all against the use of an infinite quantity [*Grösse*] as a *completed* one, which in mathematics is never allowed. The infinite is only a *façon de parler*, in which one properly speaks of limits." [Gauss in a letter to Heinrick Schumacher, 12 July, 1831 quoted in Dauben [1979] p. 120.]

It is a popular misconception that in the early nineteenth century mathematicians, led by Cauchy and others, expunged the actual infinite from analysis. What they did do was remove the specific notion of the infinitesimal as a non-zero magnitude of zero measure from analysis in favour of ratios; this work was completed by Weierstrass, to whom we owe the familiar

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[5] Wallis is said to have invented the number line, though he appears to have had reservations over negative numbers. In 1655 (In *Treatise on Conic Sections*) he writes, "I suppose any plane to be made up of an infinite number of parallel lines." In 1656 (*Artimetica Infinitorium*) he uses $\frac{1}{\infty}$ in area calculations. In the quotation there is implicit the assumption that a real line is composed of an infinite number of points of magnitude $\frac{1}{\infty}$. The width of the line is $\frac{1}{\infty}$, which is the magnitude of a point, and intersecting parallel lines gives points on a line.

[6] While the term *infinitesimal* is specifically reserved for a non-zero magnitude of zero measure, I contend that there is a second notion of infinitesimal of a zero magnitude (of necessarily zero measure). This is enshrined in the notion of a point. This notion is no less paradoxical – or should I say "quizzical" – that the other. It clearly involves a notion of complete, actual infinity.

$\varepsilon - \delta$ formulation. However, the Axiom of Completeness explicitly uses the actual infinite, and Bolzano, Weierstrss, Cauchy, Dedekind and Cantor (rightly) could not envisage the development of analysis without it.

Thus the infinitesimal, in its *other manifestation* as idealized point of no magnitude and no measure, and the actual infinite as the foundation of the Axiom of Completeness were never removed from mainstream analysis; on the contrary, nineteenth century analysis made progressively greater and greater commitments to it. And lest this conclusion be doubted, I refer to no less an authority than Hilbert. His paper *On the Infinite* [Hilbert [1925] was written in honour of Weierstrass, whom he praises for his achievement in ridding mathematics of confused notions of the infinitesimal. Yet, he remarks, "In his foundations for analysis, Weierstrass accepted unreservedly and used repeatedly those forms of logical deduction in which the concept of the infinite comes into play, as when one treats of *all* real numbers with a certain property or when one argues that *there exist* real numbers with a certain property." (In Benacerraf and Putnam [1964] p.183, Hilbert ) Thus, Weierstrass uses the notion of an infinite totality.

It is well known that the debate reached a tipping point after the discovery of the paradoxes in naive set theory – for example, Russell's paradox.

### 4.6 Russell's paradox

The Axiom of comprehension is based on the assumption that any (well-formed) formula defines a collection called a set:

<u>Axiom of comprehension</u>

Let $\varphi(x)$ be any formula, where $x$ is an indeterminate variable. Then $(\exists\alpha)(\forall x)(x \in \alpha \Leftrightarrow \varphi(x))$ where $\alpha$ is not free (that is, does not appear) in $\varphi(x)$.

In this axiom $\alpha$ represents a set. This is an *axiom schema* since it permits the formation of an infinite number of axioms, by substituting any formula for $\varphi(x)$. The paradox is obtained by substituting $\varphi(x) \equiv \neg(x \in x)$; this is, a formula defining a set as the set of all objects that are not members of themselves. This then gives, $(\exists\alpha)(\forall x)(x \in \alpha \equiv \neg(x \in x))$. By substituting $x = \alpha$, then $\alpha \in \alpha \equiv \neg(\alpha \in \alpha)$. Hence, $\alpha \in \alpha \wedge \neg(\alpha \in \alpha)$, which is a contradiction.

During the 1890s Hilbert rigorously axiomatised geometry. The success of his system laid down a new canon of mathematical practice. The great mathematicians of the nineteenth century, while devoted to rigorous proof, were more generally intuitive thinkers as well as universalists. The success of Hilbert's work reinvigorated the axiomatic method.

Hilbert agreed with the finitists that there is no justification in the facts of nature for the completed infinite. He adheres to an atomist theory of space and states, “the infinite is nowhere to be found in reality”. (Hilbert [1925] p.191) Yet he expresses his admiration also for Cantor’s theory “who systematically developed the concept of the actual infinite”. Although he describes the paradoxes that lead to the reaction against Cantor’s work, emotionally he is strongly committed to Cantor’s theory, stating, “No one shall drive us out of the paradise which Cantor has created for us.” In his view the actual content of mathematics concerns the finite symbols 1, 11, 111, 1111, ... Although “these numerical symbols which are themselves our subject matter have no significance in themselves” (Hilbert [1925] p.192) he claims that every other mathematical symbol is an abbreviation of these symbols or an operation performed on them. His solution is to claim that infinite totalities are *ideal elements*. He gives two examples of the fruitful introduction of ideal elements into mathematics: ideal elements in [projective] geometry; complex numbers as ideal elements “to simplify theorems about the existence and number of the roots of an equation.” (p.187) “... Generalizing this conclusion, we conceive mathematics to be a stock of two kinds of formulas: first, those to which the meaningful communications of finitary statements correspond; and, secondly, other formulas which signify nothing and which are the *ideal structures of our theory*.” (p.196) He interprets logic in the same manner: “the formulas of the logical calculus are ideal statements which mean nothing in themselves...” (p.197)

Hilbert’s solution was to subdivide mathematics into a finite part and an ideal part. To the finite part he ascribes synthetic content grounded in finite intuition. He expressly rejects the logicist view that mathematics is logic: “... we find ourselves in agreement with the philosophers, notably with Kant. Kant taught – and it is an integral part of his doctrine – that mathematics treats of a subject matter which is given independently of logic. Mathematics, therefore, can never be grounded solely on logic. Consequently, Frege’s and Dedekind’s attempts to so ground it were doomed to failure.” (p.192) These views make Hilbert into a *kind of* Kantian. Indeed, one might argue into a *good Kantian*! – for he follows Kant not only in ascribing synthetic content to finitary arithmetical symbols but agrees with the rejection of the actual infinite. Regarding this point G.H. Hardy [1929] commented

> Hilbert’s philosophy appears indeed to be in broad outline much the same as Weyl’s, as Weyl himself has very fairly pointed out. There is the same rejection of the possibility of any purely logical analysis of mathematics: ‘mathematics is occupied with a content given independently of all logic, and cannot in any way be founded on logic alone.’ There is the same insistence on some sort of concrete, perceptible basis, for which Hilbert (with what justice I have no idea) claims the support of ‘the philosophers and especially Kant’: ‘in order that we should be able to apply logical forms of reasoning, it is necessary that there should first be something given in presentation, some concrete, extra-logical

> object, immediately present to intuition and perceived independently on fall thought .... In particular, in mathematics, the objects of our study are the concrete signs themselves.' There is, I think, no doubt at all that Hilbert does assert, quite unambiguously, that the subject matter of mathematics proper is the actual physical mark, not general formal relations between the marks, properties which one system of marks may share with another, but the black dots on paper that we see. [p.178 of Jacquette [2002]]

The essence of Kantianism lies in the doctrine that the categories structure pre-cognitive experience, and there is no indication that Hilbert's form of Kantian philosophy went in this direction; yet it is generally agreed that Hilbert was by no means a formalist in the contemporary sense of the term. "The peculiar position of Hilbert in regard to consistency is thus no part of the formalist conception of mathematics, and it is therefore unfortunate that many persons identify formalism with what should be called Hilbertism." (Curry [1954], p. 206.]

> 4.7 Definition, Hilbertism
> *Hilbertism* is the philosophy that mathematics is divided into two parts: a synthetic part and a part comprising ideal elements. Ideal elements are justified by their relation to the synthetic part.

Hilbert expected all valid ideal elements to be consistent extensions of the synthetic part. To conclude, Hilbert, just like Kant, ascribes synthetic content to mathematical propositions, albeit in a primary sense only to those that are *finitary*, while hoping to derive all valid non-finite propositions as consistent extensions of the finite part. Hence, Wittgenstein's *attack on Hilbert* in his *Remarks on the Foundations of Mathematics*.

> But Wittgenstein's deliberate focus on the mathematical significance of Gödel's proof was grounded in the argument that, in so far as Hilbert's hierarchical conception of meta-mathematics is illicit – i.e. there are no metamathematical propositions in the manner conceived by Hilbert – Gödel's theorem has no *epistemological* significance and the coherence of Gödel's platonist gloss on the notion of mapping is dubious. For if we could perceive and describe isomorphisms between *higher* and *lower* systems then the route to a metaphysical interpretation of Gödel's theorem would indeed be unobstructed...(Shaker [1988*b*] p. 192)

According to Wittgenstein Hilbert's error was to ascribe extraneous content to mathematics. This also accounts for the short-shrift Wittgenstein gives to Gödel's theorem, which he sees as an addendum to the *falsely conceived* programme of Hilbert.

## 5 On intuitionism

Of the three philosophies emerging around 1900, namely logicism, Hilbertism and intuitionism, it would seem that intuitionism bears closest resemblance to Kantianism and to have best claim to be its heir. Yet Brouwer explicitly rejected the Kantian theory "in which time and space are taken to be forms of conception inherent in human reason" (Brouwer [1999] in Jacquette [2002] p.270). Here he identifies Kantianism with the doctrine of static categories of space and time; he goes on to infer, "For Kant, therefore, the possibility of disproving arithmetical and geometrical laws experimentally was not only excluded by a firm belief, but it was entirely unthinkable." [p.270]. Hence he concludes, "the most serious blow for the Kantian theory was the discovery of non-euclidean geometry." [p.270].

Along will all his contemporaries Kant did not anticipate the discovery of alternative geometries of the structure of space globally based on the possibilities of (a) parallel lines not intersecting even at infinity, or (b) parallel lines intersecting at more than one point. Kant made an error in not challenging the parallel postulate. For the Greeks the question as to whether parallel lines meet at infinity was *decidedly not given to intuition*. That is why they made it into a postulate and instituted the 2,500-year research programme to discover a derivation of the parallel postulate from the other axioms of geometry, which they contended *were given to intuition*. The discovery of alternative models of global space by Gauss, Lobachevsky and Bolayi put an end to this research programme by proving that the derivation was impossible. Therefore, the discovery of non-euclidean geometry by no means refutes Kantian philosophy even in its original form as the doctrine of static categories. Regarding Brouwer's own solution to the problem of the foundation of mathematics: -

> However weak the position of intuitionism seemed to be after this period of mathematical development, it has recovered by abandoning Kant's apriority of space but adhering the more resolutely to the apriority of time. This neo-intuitionism considers the falling apart of moments of life into qualitatively differing parts, to be reunited only while remaining separated by time as the fundamental phenomenon of the human intellect, passing by abstracting from its emotional content into the fundamental phenomenon of mathematical thinking, the intuition of the bare two-oneness. This intuition of two-oneness, the basal intuition of mathematics, creates no only the numbers one and two, but also all finite ordinal numbers, inasmuch as one of the elements of the two-oneness may be thought of as a new two-oneness, which process may be repeated indefinitely; this gives rise still further to the smallest infinite ordinal number $\omega$. Finally this basal intuition of mathematics, in which the connected and the separate, the continuous and the discrete are united, gives rise immediately to the intuition of the linear continuum, i.e., of the "between," which is not exhaustible by the interposition of new units and which therefore can never be thought of as a mere collection of units." [p.271]

As a phenomenological description of the concept of number, and its particular relation to the category of time this passage is a work of genius; as an attempt to preserve the tradition of Kantian intuitionism it is problematic. What Brouwer does is to turn intuition into an empirical faculty and thereby (a) negate the role of the categories in their relation to *all experience* whatsoever, (b) render the relation between mathematics (here just arithmetic) and reality obscure; (c) transform the knowledge of mathematical entities into a species of psychologism. It may well be that my concept of number is abstracted most intimately from reflection on "the falling apart of moments of life into qualitatively differing parts" in time, but this in no wise does justice to the manner in which number, as a *faculty* of the mind to structure experience, is built into the very warp and weft of all human experience of nature. Indeed, one can summarise this objection to Brouwer as the heir to Kant by saying that any theory of mathematics that *does not mention imagination as a transcendent faculty of the mind to render experience intelligible to us* must be incomplete as a Kantian theory. Thus Brouwer did a huge disservice to the Kantian tradition by distorting it into the appearance of a form of psychologism. We have also seen that he undermined Kant's philosophy of mathematics specifically by exaggerating the negative impact of the discovery of non-euclidean geometry on its validity.

Brouwer's intuitionism also misrepresents the fundamental nature of the *freedom* of *concept formation* in mathematics, which is illustrated by this quotation from Cantor: -

> Because of this extraordinary position which distinguishes mathematics from all other sciences, and which produces an explanation for the relative free and easy way of pursuing it, it especially deserves the name of *free mathematics*, a designation which I, if I had the choice, would prefer to the now customary "pure" mathematics. (Quoted in Dauben [1979] p. 132 from Cantor 1883c.)

*There is no constraint whatsoever on the formation of mathematical concepts.* In its *free aspect*, then, we may introduce into mathematics *any entity whatsoever*, be that an actual infinity or an idealised point, a complex number or a pair of parallel lines. To say that mathematics is free is to say that it is free in conception; it is not to say that it is free in justification. To justify a mathematical theory one must trace back the concepts of the theory to both particular experience and to the categories that may themselves be under dynamic transformation [explained below Sec. 6] that render all experience whatsoever intelligible. Brouwer conflates concept formation (context of discovery) with concept justification; observing that the formation of a concept from the primary intuition of "two-oneness" is justified, he concludes that no other concept can be either formed or justified in any other way. Therefore, Brouwer's fundamental error is not to trace back mathematical truth to the "intuition of two-oneness" but to do so *exclusively* and without regard to any other category or experience. He takes the particular for the universal and fails to provide an inclusive philosophy of mathematics. Thereby, he misrepresents the nature of *logic in general* whose subject matter is *the relation of the dynamic categories to experience in the field of all knowledge whatsoever.*

The spirit of our age is positivist and empiricist. Brouwer's psychologism is in marked opposition to this spirit and his philosophy in its primary aspect (illustrated above by the doctrine of the "intuition of two-oneness") has never been universally popular. Over the last century Brouwer's psychologism evolved along three lines, all of which turned it into a philosophy more and more conformable to the dominant formalism. These are all constraints on the freedom of mathematics.

1. Intuitionists reject the law of excluded middle.
2. Following the Aristotlean tradition they reject the actual infinite.
3. They insist that all mathematical entities must be constructible.

The first of these has transformed intuitionism from a philosophy of mathematics into a formal theory of many-valued logics. The other two transform it into a form of finitism. These developments taken together have turned intuitionism from a philosophy with the potential to challenge formalism into a scion of it. Constructivism and finitism, taken together, represent the philosophy of mathematics that is most conformable to strong AI, which is a philosophy of mind. Thus intuitionism has been transformed into that particular theory of mathematical concept formation and justification that conforms most closely to the metaphysical thesis of strong AI.

## 6. Static and dynamic categories

Kant introduces a distinction between the world of phenomena and the world of noumena – that is to say, between the phenomenal world of experience directly given to us, and the transcendental world of objects-in-themselves that are not given directly to us. This distinction has proven difficult for contemporary people to either comprehend or accept. It has also been subjected to intense philosophical critique. Therefore, let me attempt instead to recast this distinction in a manner that seems perfectly natural, appeals to common sense, and is consistent with anyone's experience of life in general.

Life exhibits a progress from relative unconsciousness to relative consciousness. For example, in common with most people, I recollect very little of my early childhood. There was a time when I knew little of science and mathematics as subjects, and my knowledge of the world and the information that I entertain about the world has increased with the development of my consciousness over time. Nonetheless, I can recollect that even as a child my world was relatively coherent as experience. It seems very natural to assume that, when I was born, my unconscious mind already possessed faculties by means of which my experience was structured and interpreted for my benefit, and that my conscious knowledge of those structures evolved over time as I learned to reflect carefully on what was originally put there by this unconscious power, whose origin may be material for aught I know. The contrary of this theory is the *tabula raza* – the claim of Locke's that we are born wholly without faculties and all knowledge is subsequently constructed by association[7]. It is perfectly consistent to deny the doctrine of the *tabula raza* without delving into the distinction between phenomena and noumena. One may also deny it, and still remain both a materialist and empiricist.

### 6.1 Definition, pre-cognitive stage

> By *pre-cognitive stage* of development I refer to a state of mind that is associated with early childhood, just out of infancy, between the ages approximately of three and nine. At this stage of development we are not "unconscious" but rather dimly conscious, aware and capable of forming permanent memories. Our experience of the world at this stage is already coherent and reflective of understanding, but the higher powers of analytical reflection are not developed. Hence, this is a period of *pre-cognitive* development. I associate cognitive development with analytical reflection on the nature of experience. In terms of Piaget's work in cognitive psychology the pre-cognitive stage corresponds to his *operational* stages.[8]

Kant made us aware that *all consciousness is equipped with an intensive character*. That is to say, consciousness varies in intensity. From second to second our awareness fluctuates and life generally exhibits a growth of consciousness, from a state of relative unawareness to a state of relative greater awareness. This belongs to the phenomenology of consciousness. By way of illustration, the reason why objects appear to get smaller as we get older is not that we have grown in height (for we are fully grown from about the age of sixteen) but because of our increase in awareness; space seems to contract with it. Likewise, subjectively, time seems to flow faster. This is because we are more aware in the mature adult stages of development than in the child, adolescent and early adult stages. These phenomena are illustrations of the general rule that as our concepts alter with the evolution of consciousness, so our experience of the world is altered.

When one examines Kant's *Critique of Pure Reason* in order to discover what specifically he regards as these "fundamental concepts" or "categories" one concludes that he means by them – *whatever underpins Aristotlean logic*. It is a well-established view that by formal logic Kant entertained nothing more or less than Aristotlean logic, which he regarded as a complete and finished science; he also regarded it as objectively given that the space we inhabit is Euclidean space. We may characterise Kant's interpretation of both logic and mathematics as a theory of *static categories*. The categories are given, once and for all, and there is no arguing with them. This bears a fruit of dubious benefit: all logical and mathematical knowledge is certain. This is of doubtful benefit, because, under this theory, these truths are only certain because *we have no power to think otherwise*.

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[7] I contend that Locke's theory is self-defeating, for the power of association must surely be an innate faculty, and hence a category. It would seem that the question is not whether there are categories at all, but how many of them there are, and what among them is "fundamental".

[8] For an introduction to Piaget's cognitive psychology, see Gross [1992] p. 738 et seq.

### 6.2 A neo-Kantian philosophy of dynamic concepts

In contrast with the view of Kant, I regard no category as an immutable source of truth and certainty. 1. Categories do not organise experience in such a way as to leave every question that we might ask about experience determined. Consider the category of space. Space would be structured for me in some way even if I had no conscious knowledge of geometry as a science; but this pre-cognitive structure does not determine all the properties we have come collectively to attribute of space. For example, *idealised points* are not given in a rudimentary consciousness of space. I might say that space comprises volumes enclosed within volumes, yet I cannot infer from an elementary consciousness of space that space is uniform, or that laws of congruency apply to it. The uniformity of space is a theory of space. On that basis I cannot say that *parallel lines meet at infinity* – for wherein do I have, as a consciousness still in its infancy, an experience of *the infinite extension of parallel lines*? As I grew older, my whole conscious superstructure of concepts was built over a primitive structure of fundamental concepts, even as those fundamental concepts were brought more and more into the focus of my consciousness. The study of how we in fact arrive at such superstructures is *logic in general*. Furthermore, this *general logic* is very different in kind from the specific *analytic forms of logic* that we meet in *formal*, that is, *mathematical logic*. Another fundamental category is that of *number*, which is closely related to the category of *unit*. The mind unconsciously organises experience in such a way that it divides the word into entities perceived as distinct from each other – we experience a world of *individuals*. It is a correlative of this that these individuals may be numbered, that is, counted. Thus, *number* is implicit in our pre-cognitive experience of the world, whether we can consciously count or not. *Number* is primarily a power of the mind to structure experience. Upon this unconscious power we abstract and build the conscious concept of number.

2. As we evolve, both as individuals, and as species and cultures, the validity of our categories does come into question. The question of what our sense-experience tells us about "the world" is ever open. Furthermore, *we may even deny the evidence of our own eyes*, though usually we do not do so without "good reason". For it may be that upon some consideration men will come to regard even the division of the world into distinct objects, so self-evident to sense-perception, as merely arbitrary – we may come to think that, *contrary to appearances*, all things belong to one amorphous substratum, spiritual or physical. There is no *a priori* reason to conclude that there is any constraint on mankind's ability to form concepts after the fact, and that even the categories can be questioned. About this aspect Frege wrote: -

> Space, according to Kant, belongs to appearance. For other rational beings it might take some form quite different from that in which we know it. Indeed, we cannot even know whether it appears the same to one man as to another; for we cannot, in order to compare them, lay one man's intuition of space beside another's. Nevertheless, there is something objective in space all the same;

> everyone recognizes the same geometrical axioms, even if only by his behaviour, and must do so if he is to find his way about in the world. What is objective in it is what is subject to laws, what can be conceived and judged, what is expressible in words. What is purely intuitable is not communicable.[9]

3. We have good reason to conclude that all knowledge and experience whatsoever is mediated by concepts. Conceptual knowledge affects and changes perception. *Every illusion indicates that the categories are not static.* We know that the whole essence of an illusion is to *change the appearance of things in accordance with a concept.* What I see is what I think I see! Such conclusions are the daily meat and bread of cognitive and gestalt psychologists.

Therefore, experience teaches us that we require a theory of dynamic concepts, one that sees even the most fundamental among them, those directly related as categories to some unconscious power to organise "pre-cognitive" experience, as subject to potential revision.

As it may be objected that this dynamic theory assumes freewill, let me briefly consider this point. Freewill, in the Kantian sense[10] could provide one justification for the assertion that we have the power to overthrow any of our concepts; but this view is clearly consistent with material determinism as well. The reason for this is that, as material organisms, we are subject to random extraneous influences that may alter our psychology. Biology tells us that mutation of our genes is possible, and how can we deny, save by empirical study, what the effect on the function of the brain may be by the incidence of cosmic ray upon a neuron? Whether genius is the product of freewill or of random material forces, it seems that its existence cannot be ruled out *a priori*.

## 7 The principle of harmony

What is truly primitive in the concept of number is probably no more than the bare idea of counting, but the superstructure of concepts built over that notion has tended to exhibit a process of accumulation rather than revision. The Romans did refuse to accept zero as a number, and the combination of number with the categories of the *infinite* and *the continuum* has produced a debate that has raged from the time of the Greeks to this day, and is likely to continue for as many more millennia. Nonetheless, let us agree that the category of number is relatively stable, and seek to explain that stability rather than question it.

Hume recognises only one unconscious power or category: the faculty of association, which he calls *Custom*. Over this foundation he tacitly acknowledges that the *idea of association* is built (that is, as a concept), and he also develops this into a *canon of reason*.

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[9] This is quoted by William W. Tait in his essay *Frege versus Cantor and Dedekind on the Concept of Number* in Jacquette [2002] p. 46,

[10] "By freedom ... in its cosmological meaning, I understand the power of beginning a state [spontaneously]. Such causality will not, therefore, itself stand under another cause determining it in time, as required by the law of nature." (Kant [1982] A 533, B561, p. 464. Antinomy of Pure Reason Section III: Solution of the Cosmological Totality of the Derivation of Cosmical Events from their Causes.)

> … Here, then, is a kind of pre-established harmony between the course of nature and the succession of our ideas; and though the powers and forces, by which the former is governed, be wholly unknown to us; yet our thoughts and concepts have still, we find, gone on in the same train with the other works of nature. Custom is that principle by which this correspondence has been effected; so necessary to the subsistence of our species, and the regulation of our conduct, in every circumstance and occurrence of human life. (Hume [1975], §44, p.54.)

Hume anticipates the theory of natural selection. While his interpretation of what constitutes the unconscious power of the mind is restricted in contrast to that of Kant, he clearly identifies the principle that would lead us to suppose that our primitive categories provide us with broadly correct interpretations of the reality we inhabit. Categories ensure our survival, and were they inaccurate reflections of reality, we would be poorly adapted to confront *nature red in tooth and claw*. Rationalists substitute Deity for Nature in this argument, and claim that the "pre-established harmony between the course of nature and the succession of our ideas" is the product of Divine Benevolence. Thus, both theism and atheism alike provide comfort by showing us that, in accordance with this *Principle of Harmony*, we have no particular reason to doubt the validity of our primitive categories, and if, as a result of scientific enquiry, we do doubt them, then that is an *advance* and a *gain*, and not a manifestation of failure.

> 7.1 Definition, Principle of Harmony
> There is a correspondence between the categories and reality.

The correspondence is established either by Nature or by God, and predisposes us to believe that our categories are true reflections of the structure of reality. Let us compare this theory of knowledge with one that bears some similarities to it, while being significantly different in others.

From the "biological point of view" the categories are an essential adaptation for survival; if the mind was restricted to analytical interpretations of the material presented to it, then the organism would be poorly adapted to its environment; the mind must make guesses about reality at the unconscious level, so that the organism can better survive; thus, even in the absence of any spiritualist interpretation of life, natural selection would select those organisms that are equipped with synthetic structures of reasoning. There is a kind of divinity in nature such that natural selection is in ultimate harmony with the dignity of man, and that, *were there no God to direct evolution,* nature herself in his absence would strive to select, in the course of evolution, only the most moral and most intellectually equipped representatives of humanity. The most human human is the one to strives most to understand his environment.

Let us examine this principle of harmony in relation to a model of empirical knowledge advanced by Quine: -

> The totality of our so-called knowledge or beliefs, from the most casual matters of geography and history to the profoundest laws of atomic physics or even of pure mathematics and logic, is a man-made fabric which impinges on experience only along the edges. Or, to change the figure, total science is like a field of force whose boundary conditions are experience. A conflict with experience at the periphery occasions readjustments in the interior of the field. Truth values have to be redistributed over some of our statements. Reevaluation of some statements entails reevaluation of others, because of the logical interconnections – the logical laws being in turn simply certain further statements of the system, certain further elements of the field. Having reevaluated one statement we must reevaluate some others, which may be statements logically connected with the first or may be statements of logical connections themselves. But the total field is so underdetermined by its boundary conditions, experience, that there is much latitude of choice as to what statements to reevaluate in the light of any single contrary experience. No particular experiences are linked with any particular statements in the interior of the field, except indirectly through considerations of equilibrium affecting the field as a whole. (Quine – Two dogmas of experience, in Quine [1980], p.42.)

Quine's model of the collective superstructure of all theories as a bundle of "logically connected" statements that impinges on experience at the periphery is similar to the one advocated here. Nonetheless, in his version, at the *core* of the superstructure there are postulates adopted only by convention; he recognises no force in this respect *other than experience*, and hence his sceptical conclusion that scientific theory is underdetermined. But in agreeing with Kant, I maintain that our knowledge of the world interfaces with it *in two directions*; one through sense-experience, and the other through the categories, which give rise to fundamental concepts whose validity is presumed on the basis of the principle of harmony. Like Quine, I allow that even those fundamental concepts may be dynamically overturned by evidence. Yet how this happens is the subject of general logic; it is logic, not experience per se, that glues our superstructure of theory and concepts together.

> **7.2 Definition, Quine's model**
> The totality of our knowledge or beliefs is a field [that is, structure] whose boundary conditions are experience.

I exclude from this definition the thesis that the field of knowledge is underdetermined and maintain the thesis that the field is additionally determined by the categories that impinge upon it though not at the boundary.

We are conscious beings whose humanity is quintessentially expressed in our strivings to know and understand both ourselves and the world; we are impelled both by experience and by inner need to constantly modify our concepts. These concepts are by no means disconnected from each other; they are combined by a *general logic* whose principles are active, even if they are obscure; at the pre-cognitive stage of development nature itself provides a seemingly coherent construct, albeit one that, on the *growth of consciousness*, we feel to be insufficient. We confront the world with concepts; we sense their inadequacy; through conscious endeavour we strive to develop a more cohesive superstructure of ideas; it is conscious work; we toil; we share our struggles with members of our community; we communicate with them; we debate with our opponents; we engage in the dialectical activity of weighing thesis against antithesis; we struggle for a synthesis; and in everything we do, we are conscious.

It is consistent, if, as atheists, we conclude that natural selection chooses the fittest for survival; the more conscious we are the more capable we are of solving problems. Consciousness is useful – more to the survival of the species than to the individual, but to both. If theists, then we conclude that God in his Divine Benevolence has equipped us with consciousness so that we may know ourselves and understand nature. Either way, it makes no sense to *eliminate consciousness* from the account of how we know.

## 8 Some categories

The potential and actual infinite express differing aspects of our concept *infinity*; they are related but different categories. The common notion in both is that of the *countably inexhaustible* – that if we have an infinite collection and start to enumerate or count its members we can never finish. Weyl is right, "inexhaustibility is essential to the infinite." (Weyl [1994] p.49). What is different to the two notions is that the actual infinite is bounded, the potential is not. Here we mean, "bounded in conception" for we are no more able to give a complete physical perception corresponding to a bounded infinite collection than we are able to exhaust counting its members. It is this lack of precise correspondence between physical perception, what we can see, and mental conception, which is what we can conceive, that is the root source of those legitimate doubts about their validity as concepts. Of course, a moment's reflection will show that is it a *category error* to search for a physical perception to correspond to either primitive category. Nonetheless, these categories are related in some way to perception, but differently. The category of the potential infinite is related to the category of natural number and the perception of objects as individuals, that is as distinct and determinate units; hence the potential infinite arises through counting. The category of the actual infinite is related to our perception of space and time, and in particular our perception of the continuum; it is most intimately connected to our awareness of the *flow of time* as experienced in the indeterminately bounded *moment now*. These are the legitimate and original sources of our experience of these two categories as originally presented to consciousness at the pre-cognitive stage. Of course, there is very little supra-consciousness of these categories, I mean as concepts; this supra-consciousness develops and is associated with the cognitive and mature stages of human development. Nonetheless, the coherent experience of the world that we have at the pre-cognitive stage embodies the categories:

*Unit, whole, individual*
*Now, space*

These were woven by Imagination unconsciously into the fabric of our coherent world, and are chiefly responsible for that coherence that renders reality intelligible to us at that stage. As cognition grows in us, we learn to count, and very soon the category *countable number* or *natural number* is adjoined to the category *unit*. But we cannot conceive of *counting* save as *counting in time* (the intuition of two-oneness); and likewise, we cannot distinguish the *moment now* with its continuum *space* save in relation to *time*, for all space in the moment now *flows* in the sense of flowing through subjective time. At the pre-cognitive stage *time* is arguably the more fundamental category than either *unit* or *space*, though of course in experience all three are bound together by Imagination in one unity of consciousness.

*Time, flow*

Time here is what we call *subjective time*; it is time presented to consciousness of The Self. We might say, *time* + *unit* = *number* (potential infinite). The categories are the true building blocks of the Liebnizian universal characteristic. By combination of categories we obtain new categories.

*Number, potential infinite*

The primal intuition of space is of a *totum without parts*, as a single undifferentiated continuum that is the correlative of perception. It is what I see when I look with my eyes upon the world.[11] When we identify parts (that are wholes, units), our space becomes differentiated as a *totum embracing parts* and the *totum* is linked to the category of *unit, individual. Space* has been transformed for us into the category of indefinitely bounded totum with inexhaustibly many parts. This is the continuum or actual infinite. We may say, *Now (space)* + *unit* = *continuum* (actual infinite). We obtain the category: -

[11] As Kant writes: "Space should properly be called not *compositum* but *totum*, since its parts are possible only in the whole, not the whole through the parts. ... a point is possible only as the limit of a space, and so of a composite. Space and time do not, therefore, consist of simple parts." (Kant [1982] Second Antinomy – On the Thesis. A440/B 468. p.404) "Space, however, is not made up of simple parts, but of spaces." (Kant [1982] Second Antinomy – Antithesis. A 434 / B 462. p. 402).

*Continuum, actual infinite*

By like process we can see that the category *real number* (or *point*) is adjoined to the category *actual infinite*, but as that involves the category of *counting*, then *real number* (*point*) involve in conception a *fusion of the two categories of potential and actual infinite. Point* (real number) = *potential infinite* + *actual infinite.*

*Point, real number*

The concept *point* reflects a very developed cognition, and does not belong to the pre-cognitive stage of development. We should call it a *concept* rather than a *category*. It comes much later in the evolution of consciousness in life and by the time it makes its appearance the concept *consistency* has also arisen in us. The concept *point* is a concept arising in our cognitive development and does not correspond to any concrete feature of our perception either at the pre-cognitive stage or at any later stage. No one has ever seen a point.[12] As a rule, in all our concrete experience of (empirical) reality, the categories active at the pre-cognitive stage remain strongly determining at all later stages. However, this cannot be set up as an unswerving rule, for it is also true that *concepts can and do alter perception*; so that even the concrete experience of reality may be altered by the evolution of concepts. This is most clearly seen in the speeding up of subjective time, to which I have already referred; I attribute this phenomenon to increase of consciousness; and suggest, if in old age, as it is frequently said, we slip back into an inverted childhood, then subjective space and time will expand correspondingly in that state of mind. Another point to emphasise about the pre-cognitive stage is that everything in it is consistent for us because it is coherent. Imagination has unconsciously built for us an intelligible world corresponding to concrete experience and whatever is intelligible in concrete experience is also perceived as consistent.

*Intelligible, the Self*

The intelligible is the work of Imagination, as are all the categories. The first and most fundamental way in which the world is rendered intelligible is by the presentation to consciousness (the Self) of all experience within time. Hence *time* and *the intelligible* (the Self) are fundamental. Since all the categories merge and flow into one intelligible whole, which is the Self, we may say that no category can be differentiated from another, save in thought, and by mediation of another category. How this all happens, in the sense of what its mechanical or material substratum is, if indeed it has one, is currently beyond our powers of explanation.

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[12] For an attempt to prove the existence of atoms of perception we have this from Hume: "Put a spot of ink on paper, fix your eye upon that spot, and retire to such a distance, that at last you lose sight of it; 'tis plain, that the moment before it vanish'd the image or impression was perfectly indivisible." (Hume [1985] Book I. Part II. Section II – Of the other qualities of our ideas of space and time. p.34.)

The concept of *consistency* has multiple origins. One, we have seen, is consistency in space; what is intelligible in concrete perception of objects in space entails that *no part shall be greater than the whole*, that *an object cannot be simultaneously contained in disjoint subspaces*, and so forth. This consistency is concretely manifested in our very experience of space. It is likely that the consistency of space is the root of all our notions of consistency. But we also have five senses and these can produce conflicts of another kind. It is very much the work of our Imagination at an early stage of the evolution of our categories, to construct for us a single concept of space that involves the coherence of both the visual and the tactile. At first, there is no reason to suppose these two cohere[13]. The child must learn *distance*; *uniformity of measure* is a concept that arises only much later in us, which it is the duty of education to facilitate. Through experiments in walking, touching and so forth the child actively learns to construct a concept of space in which perception and physical sensation are coherently combined; also proprioception must be essential to this *work* as inner awareness of coordination of muscle movements and other inner states is involved in constructing the concept of *distance*. (What is distant is what it takes *effort* and *coordination* to reach.) We also have the work of coordinating the information of the other senses – for the loving words of tender parents must have its origin somewhere in the visible field. But sounds that appear to come from nowhere constitute a form of inconsistency, so the category of inconsistency can be extended from its origin in spatial awareness.

By the time the concept of *real number* or *point* has arisen us, we also have the working category *consistency* and from the combination arises the *possibility* of *contradiction*. At some stage the categories of *potential* and *actual* arise in us, and thereby give rise to possible combinations. We can see that in the evolution of cognition (understanding) the categories have the power to combine and multiply among themselves, so that it would be almost possible for a cognition to evolve without the intervention of Experience, but of course, Experience itself is the universal stimulus to this development, and a fully developed theory of the evolution of cognition, which is the genesis of our categories, could not be complete without it[14].

### 8.1 Analytical logic

Modern formal logic is a more rigorous expression of the syllogistic logic of Aristotle. We have seen [Chap. 4 Sec. 2] that idea of syllogistic logic arose from an *analogy* with containment in space, and is developed from the relation of part to whole. Hitherto, only one kind of logic has ever been considered formally, and herein we grasp its essence – as the

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[13] This is confirmed by the empirical studies of Piaget. At the sensorimotor stage of development – the very earliest stage – the child is said to have almost no concept of distance.

[14] It can be seen that, as Kant, I am adopting a middle ground between rationalism and empiricism, and seek, not to give unswerving support to either side, but to find what is right in both and combine them into a single theory.

*analytic logic that is based on exact analogy with the spatial relation of part to whole*, which we may now distinguish from *synthetic logic*, which is logic of exact reasoning that is not based on an analogy with a spatial relation. We see also that the underlying pattern of all logic whatsoever, analytic or synthetic, is analogy.

## 9 Analogy

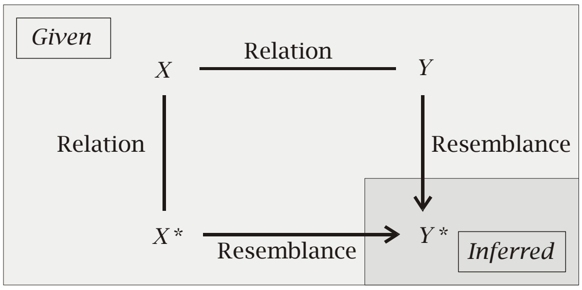
We understand intuitively what is meant by an *analogy*, but the time has come to clarify the meaning of this all-important term and to explain what the distinction between an *exact* and *inexact* analogy is. For an illustration of argument by analogy I take the following extract from Hume's *Dialogues Concerning Natural Religion.*

> *Cleanthes is defending the argument from design, and Philo, who is speaking in this extract, is subjecting it to critical scrutiny. A third member of the party is Demea, who represents the old dogmatic view of theology, and defends the medieval proofs of the existence of God such as the cosmological argument. Demea does not speak in this extract.*
>
> That a stone will fall, that fire will burn, that the earth has solidity, we have observed a thousand and a thousand times; and when any new instance of this nature is presented, we draw without hesitation the accustomed inference. The exact similarity of the cases gives us a perfect assurance of a similar event, and a stronger evidence is never desired nor sought after. But wherever you depart, in the least, from the similarity of the cases, you diminish proportionately the evidence, and may at last bring it to a very weak *analogy*, which is confessedly liable to error and uncertainty. After having experienced the circulation of the blood in human creatures, we make no doubt that it takes place in Titius and Maevius; but from its circulation in frogs and fishes it is only a presumption, though a strong one, from analogy that it takes place in men and other animals. The analogical reasoning is much weaker when we infer the circulation of the sap in vegetables from our experience that the blood circulates in animals; and those who hastily followed that imperfect analogy are found, by more accurate experiments, to have been mistaken.
>
> If we see a house, Cleanthes, we conclude, with the greatest certainty, that it had an architect or builder because this is precisely that species of effect which we have experienced to proceed from that species of cause. But surely you will not affirm that the universe bears such a resemblance to a house that we can with the same certainty infer a similar cause, or that the analogy is here entire or perfect. The dissimilitude is so striking that the utmost you can here pretend to is a guess, a conjecture, a presumption concerning a similar cause; and how that pretension will be received in the world. I leave it to you to consider. (Hume [1779])

This passage illustrates what is meant by analogy and also *weak* analogy. Hume's examples are drawn from the empirical realm, where he shows that reasoning by cause and effect rests entirely on analogy. Analogy is also the foundation of the non-empirical science of mathematics and of reasoning in pure philosophy.

### 9.1 Definition, analogy

An argument by *analogy* is any inference from something given to something not given on the basis of on a double resemblance derived from two relations.




Here *relation* and *resemblance* are primitive notions. As every relation bears resemblance to every other instance of that relation, they belong to the same category.

From Hume's examples, let $X$ = man, $X^*$ = circulation, $Y$ = frog, $Y^*$ = circulation. This is a case of an inexact analogy. In an exact analogy we are recognising another instance of a relation and the range of a relation is extended. In an inexact analogy there are two distinct relations between which we perceive some similitude but allow for the possibility of dissimilitude.

### 9.2 Definition, exact resemblance

Let $R$ be a relation between $X$ and $Y$ and $R^*$ be a relation between $X^*$ and $Y^*$. Then the resemblance between $R$ and $R^*$ is said to be *exact* if $R$ and $R^*$ are instances of the same relation. (Formally, we may write $R = R^*$ and $XRY \equiv X^*R^*Y^*$.) This makes $X^*$ and $Y^*$ into *substitution instances* for $X$ and $Y$ and conversely.

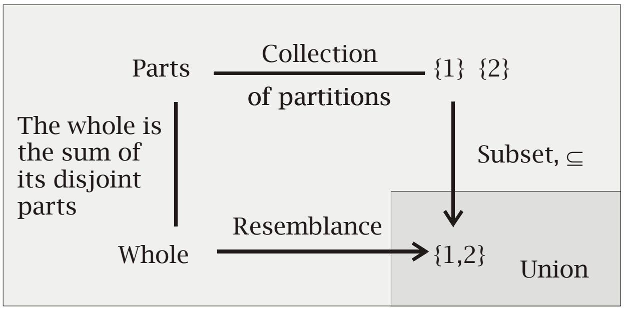
### 9.3 Definition, exact analogy

An *exact analogy* occurs when both resemblances in the analogy are exact. It may also be said to be *perfect.* An *inexact analogy* is an analogy where one or both resemblances are not exact. Inexact analogies are attended by an intensive quality of the *degree of inexactness.* This cannot be given a determinate definition but depends on judgement. A *strong* analogy is either an exact analogy or an inexact analogy whose degree of exactitude is great though not perfect. A weak analogy is an inexact analogy with small exactitude.

In science we accept, to a degree, inexact analogies, but the more inexactitude in an analogy the less conviction it induces in us. In an exact science only exact analogies are accepted. A weak analogy is an inexact analogy such that it can only induce in us a *weak belief*, *surmise* or *conjecture*. The strength of conviction in an analogy is influenced by emotion; strong emotional commitment to an idea causes analogies that support that idea to appear more exact than they might otherwise. Hume refutes the teleological argument on the basis that the analogy that it uses is "very weak". (Hume [1779])

At the pre-cognitive stage the analogies constructed for us by Imagination are largely unconscious. The manner in which the conscious mind influences the unconscious mind is by bringing unconscious content to consciousness. We have no conscious influence over whatever remains unconscious. Hence, in the first instance, categories that are constructed by exact analogy at the unconscious level have the appearance of necessity; for literally we have no power to question the exactitude of the analogy that is presented to us and, in the main, woven by the unconscious mind into the underlying structure of our experience, particularly at the pre-cognitive stage. To defy the analogy would be to deny the evidence of our own eyes. By bringing all categories to consciousness we gain the power to question every category and so render our beliefs more conformal to the content of our experience[15].

The relation in set theory to spatial containment is an example of an exact analogy. Recall that sets are extensional, determinate collections; therefore, precisely because all intensions have been eliminated, the properties of a set are determined wholly by the primitive membership relation. The logic of sets originates in the exact analogy between the primitive membership relation and containment.




The language of spatial containment is frequently used in set theory and those theories defined in set theoretical terms. We talk of *subspace*, *subset*, *empty set*, *domain*, *point set*, and so forth[16].

[15] I conjecture that human consciousness evolves towards more and more empiricism.

[16] The whole concept of multiple dimensions is drawn from analogy from the spatial relation of surface to volume. The concept of *objective, real time* is created by analogy to the spatial relation of dimension.

### 9.4 Definition, analytic logic

*Analytic logic* is the exact science of inferences that originate in an exact analogy with the spatial relation of part to whole.

**Examples**

The following are analytic logics:

1. Syllogism.
2. Classical propositional calculus.
3. Classical predicate calculus.
4. Intutionistic logic.
5. The effective part of first-order set theory.

This list is also not intended to be exhaustive. All formal logics defined and studied in the last century or in any previous century are expected to be analytic logics. This includes modal logic.

### 9.5 Definition, inductive logic

*Inductive logic* is the science of inferences that originate in *inexact* analogies[17].

In the terminology of Kant all analytic logic has a synthetic base. Its rests synthetically on the exact analogy with space, and where that analogy fails even the application of analytic logic may fail. Thus, we see that analytic logic is limited in scope to precisely those cases where the analogy is exact. Hence, whatever follows from *within* analytic logic is *analytic a priori* in the sense of Kant, but analytic logic taken altogether is *synthetic a priori.* Any application of that logic requires a synthetic act of judgement of the mind.

### 9.6 Definition, synthetic logic

*Synthetic logic* is the exact science of inferences that originate in any exact analogy that is not based on a spatial relation of part to whole.

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[17] At the pre-cognitive stage the child's Imagination unconsciously makes much use of inductive logic to build the child's *rudimentary scientific knowledge of the world.* This would include such *inductive generalizations* such as *fire burns* and *water quenches thirst.* Science as a cognitive endeavour to build a more precise and enduring knowledge of cause and effect has a tendency to substitute exact reasoning for the inexact reasoning of inductive logic. Here I refer to the substitution of the hypothetical-deductive method for the inductive method. Note that at the pre-cognitive stage the child is largely unconscious, and has no *method* as such, the method he has being the work of his unconscious mind. At a later cognitive stage, by reflection on the operation of his unconscious mind, he may bring the process of inductive generalization to consciousness, and hence arrive at the *method of induction.* The inexactness in this and its inability to take a place in a cohesive theory encourages the conscious mind to abandon this as a formal method in favour of one that employs exact reasoning.

Arithmetic is synthetic logic. In general the following are synthetic: (1) the exact logic of philosophical discourse; (2) irreducibly second-order logic; (3) the non-effective part of first-order set theory.

### 9.7 Natural language logic

In formal logic the logical symbols, $\wedge, \vee, \neg, \supset, \ldots$ correspond to conjunctions and parts of speech of natural language: "and", "or", "not" and "if… then" respectively. In the definition of an analytic logic we have opened up the possibility that the meaning and mode of use of the natural language terms in this pairing are *not always analytic*, for analytic logic is limited in scope to precisely those cases where the analogy with spatial containment is exact. There is no a priori reason to believe that every use of the word "and" implies or rests upon an exact analogy with spatial containment. Rather "and" is used to *connect ideas*. For example:

> The doctrine that pleasure, *among other things*, is good as an end, is not Hedonism; <u>and</u> I shall not dispute its truth[18].

My underlining. The relation conjoined here is not prima facie a spatial relation. When we formalise natural language arguments by encoding them in propositional or predicate logic, we are very often *constraining* the logical connectives to behave as if they were analytic relations. Definitions in mathematics frequently go outside the confines of the first-order analytic connectives. Here is a simple instance of this: -

$p \equiv q$ if, and only if $p \supset q$ and $q \supset p$

The first-order connective, $\equiv$, which stands for the equivalence of propositions $p$ and $q$ is defined by means of the natural language version of that equivalence, *if, and only if*. Surely the meaning of *if, and only if* must be given to intuition directly? To explain this situation it is said that $p \equiv q$ lies in the *object-language* and *if, and only if* lies in the meta-language, and the purpose of the definition is to fix the meaning of the symbol in the object-language by means of discourse in the meta-language. If this is true, it saves the definition from the charge of circularity, but one would like to say that prima facie it does not encourage one to think that the meta-language is itself an instance of analytic logic. In the following quotation the meta-language is meta-mathematics and the object-language is mathematics.

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[18] This sentence was selected by choosing the first philosophy text on my shelf that came to my eye; selecting a page at random and choosing the first sentence with an "and" from the first paragraph. It comes from p.62 of G.E. Moore's *Principia Ethica*. Cambridge University Press, 1979 edition. (Original date of publication, 1903.)

> … Nothing stated above implies that in our opinion there is any fundamental difference between metamathematics and mathematics "proper". Quite the contrary: we believe that, from every reasonable point of view, metamathematics is an integral part of mathematics. (Henkin, Monk and Tarski [1971])[19]

This doctrine is essential to formalism, which does not permit logic (or mathematics) to break out of the circle of formal processes into cognitive meaning. This is the doctrine that the meta-language is also just another instance of the formal process encapsulated in the object-language.

### 9.8 Tautologies and vacuity

We have seen that the elementary symbols of classical propositional logic are $\wedge, \vee, \neg, \supset$ corresponding to the logical connectives in natural language, "and", "or", "not" and "if… then". The symbol $\supset$ is also known as material implication. As is well known we only need two of these symbols to form a complete system: $\neg$ and any one other[20]. The full system is retained because all four symbols are useful; however, for many purposes a system based on $\wedge, \vee, \neg$ is most perspicuous, because of its transparent relation to the Boolean lattice. The presence of the symbol $\supset$ gives to the tautologies that use it a kind of synthetic character that can disguise its essentially analytic and "vacuous" content

To say that all mathematics is a form of analytic logic (recalling, that by analytic logic here I encompass also the effective part of set theory) is something that I think runs against all common sense. Is it really likely that every theorem, lemma, result, proposition or valid formula of mathematics is nothing but a glorified tautology? About this problem Poincaré wrote: -

> If … all the propositions [of mathematics] may be derived in order by the rules of formal logic, how is it that mathematics is not reduced to a gigantic tautology? The syllogism can teach us nothing essentially new, and if everything must spring from the principle of identity, then everything should be capable of being reduced to that principle. Are we then to admit that the enunciations of all the theorems with which so many volumes are filled, are only indirect ways of saying that A is A? (Poincaré [1982], p.394)

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[19] The quotation comes towards the end of their Introduction to *Cylindric Algebras I*. The authors are discussion how a metalogical representation theorem leads to new forms of predicate logic.

[20] It is also well known that in fact a single symbol, the Sheffer stroke, is sufficient. But we shall have no need of that here.

## 10 The transcendental deduction

Any argument from knowledge to a conclusion about the nature of the mind may be called a *transcendental deduction*. The originator of this argument was Plato, who concluded in the *Meno* that our knowledge of mathematics gives us reason to believe, “Men’s souls are immortal. Souls pass through death and are reborn, but they are never really annihilated.” He derives this conclusion from the claim that when we learn mathematics we are actually recollecting what we learned in previous life, for, “The soul, since it is immortal, has been born many times, and has seen all things both here and in the other world. It has already learnt everything that is. So we should not be surprised if we discover that the soul can recall the knowledge of virtue or any other matter that it formerly possessed.” Another version of the same pattern of argument is offered by Plato as a refutation of empiricism: -

> SOCRATES. Now take sound and colour. Have you not, to begin with, this thought which includes both at once – that they *exist*?
>
> THEAETETUS. I have.
>
> SOCRATES. And, further, that each of the two is *different* from the other and the *same* as itself?
>
> THEAETETUS. Naturally.
>
> SOCRATES. And again, that both together are *two*, and each of them is *one*?
>
> THEAETETUS. Yes.
>
> SOCRATES. And also you can ask yourself they are *unlike* each other or *alike*?
>
> THEAETETUS. No doubt.
>
> SOCRATES. Then through what organ do you think all this about them both? What is common to them both cannot be apprehended either through hearing or through sight … through what organ does that faculty work, which tells you what is common not only to these objects but to all things – what you mean by the words “exists” and “does not exist” and the other terms applied to them in the questions I put a moment ago? What sort of organs can you mention, correspondign to all these terms, through which the preceiving part of us perceives each one of them?
>
> THEAETETUS. You mean existence and non-existence, likeness and unlikeness, sameness and difference, and also unity and numbers in general as applied to them; and clearly your question covers ‘even’ and ‘odd’ and all that kind of notions. You are asking, through what part of the body our mind perceives these?
>
> SOCRATES. You follow me admirably, Theaetetus; that is exactly my question.
>
> THEAETETUS. Really, Socrates, I could not say, except that I think there is no special organ at all for these things, as there is for the others. It is clear to me that the mind itself is its own instrument for contemplating the common terms that apply to everything. (Plato [1979] A – C. p. 184)

Since meanings cannot be copied from experience, then there must be some other source of our knowledge of them. Plato calls meanings, Ideas or Forms and he claims that Forms are abstract entities not located in space and time, and our experience of them explains how we can have knowledge of the general terms of our language.

Kant's *Critique of Pure Reason* is another attempt at a transcendental deduction – indeed, the name of the argument seems to originate with Kant. The aim is to account for the origin of *synthetic a priori* knowledge: -

> But though all our knowledge begins with experience, it does not follow that it all arises out of experience. For it may well be that even our empirical knowledge is made up of what we receive through impressions and of what our own faculty of knowledge (sensible impressions severing merely as the occasion) supplies from itself. If our faculty of knowledge makes any such addition, it may be that we are not in a position to distinguish it from the raw material, until with long practice of attention we have become skilled in separating it.

This paper supports Kant's conclusion that the mind itself, at some unconscious level, supplies fundamental concepts that he calls categories that structure, organise experience and render it intelligible to us. Kant concluded that the mind is equipped with a transcendent power; he called this power *imagination.*[21] It is a step from this to the further conclusion that human identity also transcends experience: -

> There can be in us no modes of knowledge, no connection or unity of one mode of knowledge with another, without that unity of consciousness which precedes all data of intuitions, and by relation to which representation of objects is alone possible. This pure original unchangeable consciousness I shall name *transcendental apperception.* That it deserves this name is clear from the fact that even the purest objective unity, namely, that of the *a priori* concepts (space and time), is only possible through relation of the intuitions to such unity of consciousness. The numerical unity of this apperception is thus the *a priori* ground of all concepts, just as the manifoldness of space and time is the *a priori* ground of the intuitions of sensibility.(Kant [1982] A. 107, p. 136. Transcendental Deduction A – The A priori Grounds of the Possibility of Experience. 3. The Synthesis of Recognition in a Concept.)

Elsewhere Kant calls this *transcendental apperception* the transcendental self. It is the pure consciousness that accompanies all our experiences.

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[21] … the reproductive synthesis of the imagination is to be counted among the transcendental acts of the mind. We shall therefore entitle this faculty the transcendental faculty of imagination. (Kant [1982] A.101 p. 132 - Transcendental Deduction A – The A priori Grounds of the Possibility of Experience. 3. The Synthesis of Reproduction in Imagination.)

> The abiding and unchanging 'I' (pure apperception), forms the correlate of all our representations in so far as it is to bed at all possible that we should become conscious of them. All consciousness truly belongs to an all-comprehensive pure apperception, as all sensible intuition, as representation does to a pure inner intuition, namely, to time. (Kant [1982] A.124 p. 146 - Transcendental Deduction A – The A priori Grounds of the Possibility of Experience. 3. The Synthesis of Recognition in a Concept.)

Kant replaces the *cogito ergo sum* of Descartes with the *sum*; not "I think therefore I am" but simply "I am". In every act of the mind I am aware that I am I. It would not be possible to ever bring together two distinct experiences, unless they were brought together into the same consciousness. Therefore, the transcendental self must always be assumed to be one and the same self. Kant also offers an independent argument to prove that pure transcendental apperception embraces the moment now in the synthesis of time, which is the fundamental category: -

> If we were not conscious that what we think is the same as what we thought a moment before, all reproduction in the series of representations would be useless. For it would in its present state be a new representation which would not in any way belong to the act whereby it was to be gradually generated. The manifold of the representation would never, therefore, form a whole, since it would lack that unity which only consciousness can impart to it. If, in counting, I forget that the units, which now over before me, have been added to one another in succession, I should never know that a total is being produced through this successive addition of unit to unit, and so would remain ignorant of the number. For the concept of the number is nothing but the consciousness of this unity of synthesis. (Kant [1982] A.103 p. 134 - Transcendental Deduction A – The A priori Grounds of the Possibility of Experience. 3. The Synthesis of Recognition in a Concept.)

If this argument is valid, then the transcendental self cannot be regarded as an object in time.

## 11 The transcendental deduction and Poincaré's thesis

Be this all as it may, in this paper I have also attempted a transcendental deduction of more restricted kind. It is the nature of a transcendental deduction to refute a particular conception of the mind, and to do so by showing that such and such a conception could not account for our knowledge. Here I have refuted the doctrine known as strong AI and the parallel philosophy of mathematics known as formalism. I have done so by showing that formalism as a theory of mathematical knowledge cannot account for our knowledge of proof; that strong AI, which is a metaphysical doctrine of the nature of the mind, is not consistent with mathematical knowledge. The transcendental deduction offered here is not so lofty in its conclusions as those offered by Plato or Kant: what is refuted is very specific. I cannot say, on these grounds alone, that either materialism or empiricism are false; I have no wish to, for both doctrines may evolve in time into philosophies richer and more universally satisfying than their current instances would allow. Whether the mind of man is matter or whether it is something distinct from matter, it has a kind of strange divinity about it – or at the least we may say that a certain mechanical explanation of it has been proven to be inadequate to account for it. It is too soon to relinquish the notion that there is something special about being human.